Table 3 Mode 1 eigenvalues for $a = 0.5$ ($\varepsilon^* = 0.0126$)

ε	Exact	Singular perturbation	Southwell sum
0.001	2.86	2.79	2.70
0.004	3.51	3.14	3.29
0.010	4.72	3.60	4.48
....			
0.040	10.67	5.08	10.41
0.100	22.54	7.24	22.28
0.400	81.88	15.56	81.60

Table 4 Mode 1 eigenvalues for $a = 0.9$ ($\varepsilon^* = 0.00012$)

ε	Exact	Singular perturbation	Southwell sum
....			
0.001	139	38	138
0.004	510	79	507
0.010	1,252	149	1,250
0.040	4,960	460	4,958
0.100	12,378	1,042	12,375
0.400	49,465	3,830	49,454

Since λ'_{bn} increases more rapidly than λ'_{cn} as n increases, it is apparent that ε_n^* becomes monotonically smaller as the mode number n increases. Hence, in terms of ε, the range of validity of singular perturbation theory decreases with mode number. In contrast to the criterion for validity of the singular perturbation methods employed by Eick and Mignolet, the ε_n^* criterion does not fail for the first mode when the beam is fixed at the axis of rotation. However, the magnitudes of ε_n^* are generally consistent with the ε ranges for validity of the singular perturbation method discussed by the authors.

For the cases in which the beam is fixed outboard of the center of rotation at distance a, expressed as a fraction of the tip radius R, the Southwell sum for the first mode for uniform mass and bending stiffness can be expressed approximately as

$$\lambda'_{s1} = \left[1 + \tfrac{3}{2}a/(1-a)\right] + 12.36\varepsilon/(1-a)^4 \cdots \tag{5}$$

The first term is an approximate expression for λ_c, the lowest frequency of a rotating cable attached at point a. It is an approximation to the exact eigenvalue $\lambda'_c = \nu(\nu+1)/2$ for $P\nu(a) = 0$ where $P(\nu)$ is the Legendre function or order ν (and where ν need not be an integer). Comparisons of the results from the first term of Eq. (5) with exact values indicate an error of less than 2% for $a < 0.6$ and about 3% at $a = 0.9$. These errors are on the high side since the approximation for λ_{c1} is based on a rigid rod, hinged at point a, and represents, in effect, a Rayleigh quotient for the lowest cable eigenvalue. With this approximation, the Southwell sum for the lowest eigenvalue is no longer a rigorous lower bound. The second term adjusts the beam bending frequency formula for $a = 0$ to account for the reduced beam length from the fixed point at a.

Tables 3 and 4 further demonstrate the significance of ε^* as a breakpoint for the accuracy of the Southwell sum approximation and also, in an inverse way, for the accuracy of the singular perturbation approximation, for $a \neq 0$. It is interesting that in the case of $a = 0.90$ in Table 4, ε^* is so much less than the values of ε tabulated that Southwell sums give accurate results for the lowest eigenvalues within 1% of exact values throughout whereas the singular perturbation results are wildly inaccurate for all values of ε in the table.

References

[1]Eick, C. D., and Mignolet, M. P., "Vibration and Buckling of Flexible Rotating Beams," *AIAA Journal*, Vol. 33, No. 3, 1995, pp. 528–538.

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Reply by the Authors to A. Flax

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THE comment by Flax[1] provides an interesting combined use of the singular perturbation method and of Southwell's approximation but also indirectly reemphasizes two important conclusions that were drawn in our original paper,[2] namely the high accuracy and usefulness of the regular perturbation approach and the importance of the parameter εn^2 where $\varepsilon = EI/m\Omega^2\bar{R}^4$ (see Ref. 2) and n is the mode number. These issues are discussed further below.

Accuracy and Usefulness of the Regular Perturbation Approach

It was found in Ref. 2 that the natural frequencies and mode shapes of a rotating beam can be predicted accurately for essentially all values of the normalized beam stiffness ε and clamping radius a by relying on the regular perturbation method described therein. In particular, it was shown that the eigenvalues (squares of the natural frequencies) admit a series representation of the form

$$\lambda = \left(\lambda_{-1}/\varepsilon^{\frac{1}{3}}\right) + \lambda_0 + \lambda_1\varepsilon^{\frac{1}{3}} + \cdots \tag{1}$$

where the coefficients λ_i, $i = -1, 0, 1, \ldots$ are functions of the sole parameter

$$\Delta = (1-a)/\varepsilon^{\frac{1}{3}} \tag{2}$$

Further, it was shown that the eigenvalue estimates computed on the basis of only the first two terms in Eq. (1) are extremely reliable for all clamping radii a and normalized beam stiffnesses ε investigated. This surprising result then motivated the determination of the asymptotic behavior of the coefficients λ_{-1} and λ_0 as $\Delta \to 0$ and $\Delta \to \infty$, which can be expressed as[2]

$$\lim_{\Delta\to 0} \lambda_{-1} = \left(\beta_n^4/\Delta^4\right) + (\bar{\gamma}_n/\Delta) \tag{3}$$

$$\lim_{\Delta\to 0} \lambda_0 = \delta_n \tag{4}$$

where β_n are the standard nonrotating clamped-free beam frequencies and

$$\bar{\gamma}_n = 2 - \alpha_n\beta_n + \frac{\alpha_n^2\beta_n^2}{2} \tag{5}$$

$$\delta_n = -\tfrac{3}{4} + (\alpha_n\beta_n/2)[1 - (\alpha_n\beta_n/3)] \tag{6}$$

with

$$\alpha_n = \frac{\cos(\beta_n) + \cosh(\beta_n)}{\sin(\beta_n) + \sinh(\beta_n)} \tag{7}$$

and

$$\lim_{\Delta\to\infty} \lambda_{-1} = \mu_n^2/4\Delta \tag{8}$$

$$\lim_{\Delta\to\infty} \lambda_0 = -\tfrac{1}{6}\left[1 + \left(\mu_n^2/4\right)\right] \tag{9}$$

where μ_n designates the nth zero of the Bessel function J_0.

It was observed in our original investigation[2] that the limits given by Eqs. (3–7) were valid in the range $\Delta \in [0, 3]$ while Eqs. (8)

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and (9) provided good estimates of the coefficients λ_{-1} and λ_0 for $\Delta \in [10, \infty]$ for the first mode.

The suggestion by Flax[1] to use the Southwell sum to estimate the natural frequencies of the rotating beam when the normalized beam stiffness ε exceeds a certain threshold ε^* is equivalent, in view of Eq. (2), to the use of this approximation technique for small values of the parameter Δ. On this basis, it is first desired here to compare the accuracy of the Southwell sum and the asymptotic relations, Eqs. (3–7). Shown in Tables 1–4 are the eigenvalues (squares of natural frequencies) for the first two modes with $a = 0$, and the first mode only for $a = 0.5$ and 0.9 as computed by the series solution[2] (the exact results), by the Southwell sum,[1] and by relying on Eqs. (3–7). Clearly, the limiting results given by Eqs. (3–7) provide excellent approximations of the eigenvalues and are consistently more reliable than their Southwell sum counterparts. Further, this observation is valid not only below the dotted line ($\varepsilon > \varepsilon^*$) but also for a wide range of values of ε below the threshold ε^*. In fact, it is only for very small values of ε and a, or equivalently for $\Delta \gg 1$, that the accuracy of the limiting behavior, Eqs. (3–7), decreases. In this range of values, however, the other asymptotic relations, Eqs. (8) and (9) are valid and can be used to provide the required accurate estimates of the eigenvalues.

It is then suggested that Eqs. (3–7) be used when the values of ε and a are such that the parameter Δ is small, $\Delta < \Delta_1$ say, that Eqs. (8) and (9) be relied upon when Δ is large, i.e., $\Delta > \Delta_2$, and that interpolation between the two estimates be used when $\Delta_1 < \Delta < \Delta_2$. Shown in the last columns of Tables 1–4 are the results of this approximation strategy with $\Delta_1 = 3$ and $\Delta_2 = 10$ for the first mode (see Ref. 2) and $\Delta_1 = \Delta_2 = \infty$ for the higher modes (see justification below). Clearly, the eigenvalues obtained in this manner are extremely close to their exact counterparts, consistently better than the singular perturbation–Southwell sum combination suggested by Flax,[1] provided that $\varepsilon \geq 0.001$. Finally, note that, from the standpoint of obtaining a first estimate of the natural frequencies to be used in a preliminary design, Eqs. (3–9) only require simple mathematical operations.

Table 1 Mode 1 eigenvalues for $a = 0$

ε	Exact (Ref. 2)	Southwell sum (Ref. 1)	Regular perturbation Eqs. (3–7)	Regular perturbation Eqs. (3–9)
0.001	1.07	1.01	1.21	1.04
0.004	1.15	1.05	1.24	1.15
0.010	1.25	1.12	1.32	1.25
0.040	1.66	1.49	1.69	1.69
...				
0.100	2.42	2.24	2.43	2.43
0.400	6.14	5.94	6.14	6.14
1.000	13.55	13.36	13.56	13.56
4.000	50.64	49.94	50.64	50.64

Table 2 Mode 2 eigenvalues for $a = 0$

ε	Exact (Ref. 2)	Southwell sum (Ref. 1)	Regular perturbation Eqs. (3–7)	Regular perturbation Eqs. (3–9)
0.001	6.78	6.48	6.96	6.96
0.004	8.36	7.94	8.42	8.42
0.010	11.3	10.9	11.3	11.3
...				
0.040	25.9	25.4	25.9	25.9
0.100	55.0	55.0	55.0	55.0
0.400	200.7	200.2	200.7	200.7
1.000	492.0	491.5	492.0	492.0
4.000	1948.6	1947.9	1948.6	1948.6

Table 3 Mode 1 eigenvalues for $a = 0.5$

ε	Exact (Ref. 2)	Southwell sum (Ref. 1)	Regular perturbation Eqs. (3–7)	Regular perturbation Eqs. (3–9)
0.001	2.86	2.70	2.96	2.83
0.004	3.51	3.29	3.56	3.53
0.010	4.72	4.48	4.74	4.74
...				
0.040	10.67	10.41	10.68	10.68
0.100	22.54	22.28	22.54	22.54
0.400	81.88	81.60	81.88	81.88

Table 4 Mode 1 eigenvalues for $a = 0.9$

ε	Exact (Ref. 2)	Southwell sum (Ref. 1)	Regular perturbation Eqs. (3–7)	Regular perturbation Eqs. (3–9)
...				
0.001	139	138	139	139
0.004	510	507	510	510
0.010	1252	1250	1252	1252
0.040	4960	4958	4960	4960
0.100	12378	12375	12378	12378
0.400	49465	49454	49464	49464

Importance of the Parameter εn^2

It was demonstrated in our original investigation[2] that the reliability of the singular perturbation method decreases monotonically as the mode number n increases, even for very small values of the normalized beam stiffness ε. This result was justified in particular by analyzing the magnitudes of the different terms of the equation of motion over the entire length of the beam for different values of n (see the Appendix of Ref. 2). It was then suggested that the singular perturbation technique is accurate provided that $\varepsilon n^2 \ll 1$, not simply $\varepsilon \ll 1$. The comments by Flax[1] and the preceding discussion provide two new justifications of this result.

Flax[1] accurately observed that the reliability of the singular perturbation results deteriorates rapidly as ε exceeds a certain threshold ε^*, which he estimated to correspond to the equality of the natural frequencies of the rotating cord and the nonrotating beam.

Thus, ε^* is such that

$$\lambda = \frac{\nu_n(\nu_n + 1)}{2} = \frac{\beta_n^4 \varepsilon^*}{(1-a)^4} \tag{10}$$

or equivalently,

$$\varepsilon^* = \frac{\nu_n(\nu_n + 1)(1-a)^4}{2\beta_n^4} \tag{11}$$

To analyze the behavior of this threshold as the mode number increases, note that

$$\beta_n \to (2n-1)(\pi/2) \qquad \text{as} \qquad n \to \infty \tag{12}$$

Further, assume for simplicity that $a = 0$ so that $\nu_n = (2n - 1)$ and

$$\varepsilon^* \to 2/n^2\pi^4 \qquad \text{as} \qquad n \to \infty \tag{13}$$

It is thus concluded that the singular perturbation estimates of the natural frequencies corresponding to $a = 0$ will be accurate when $\varepsilon < \varepsilon^*$, or equivalently when $\varepsilon n^2 < (2/\pi^4)$ as $n \to \infty$.

A final justification of the critical character of the product εn^2 can be obtained by relying on the asymptotic relations given by Eqs. (3–9). Specifically, it is known[2] that the singular perturbation approximation corresponds to the limit $\Delta \to \infty$ to which Eqs. (8) and (9) are associated. Thus, the range of validity of the singular perturbation method could be approximately described as $\Delta > \Delta_{cr}$, where the value Δ_{cr} is such that the leading terms in the asymptotic relations (3) and (8) are equal. This condition leads to

$$\Delta_{cr}^3 = \frac{(1-a)^3}{\varepsilon_{cr}} = \frac{4\beta_n^4}{\mu_n^2} \tag{14}$$

Considering again the behavior as the mode number increases, $n \to \infty$, it is found that[3]

$$\mu_n \to (2n-1)(\pi/2) + (\pi/4) \qquad \text{as} \qquad n \to \infty \tag{15}$$

and thus that

$$\varepsilon_{cr} \rightarrow \frac{(1-a)^3}{4n^2\pi^2} \quad \text{as} \quad n \rightarrow \infty \tag{16}$$

The estimates of the natural frequencies obtained by the singular perturbation approach will then be accurate when $\varepsilon < \varepsilon_{cr}$, or equivalently when $\varepsilon n^2 < [(1-a)^3/4\pi^2]$ as $n \rightarrow \infty$.

The closeness of the threshold values ε^* [Eq. (13)] and ε_{cr} [Eq. (16)] further demonstrates the validity of these arguments and of the condition $\varepsilon n^2 \ll 1$ for the reliability of the singular perturbation method.

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[1]Flax, A., "Comment on 'Vibration and Buckling of Flexible Rotating Beams'," *AIAA Journal*, Vol. 34, No. 3, 1996, pp. 640, 641.

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Errata

Flow Visualization Using Natural Condensation of Water Vapor

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[AIAA Journal, 33(11), pp. 2234–2236 (1995)]

THE figures were illegible. They are reproduced below.

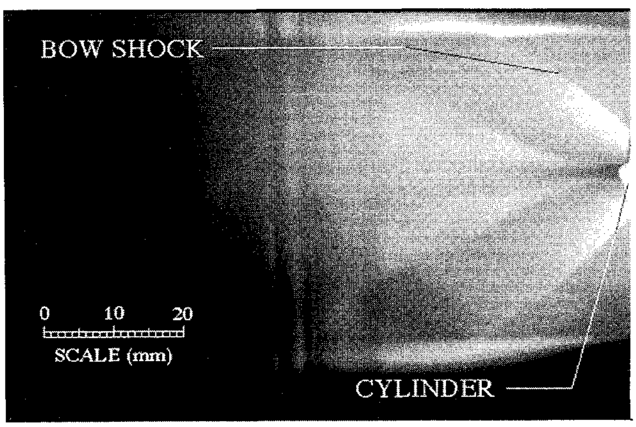

Fig. 1 Flow over a circular cylinder with Mach 2.0 nozzle.

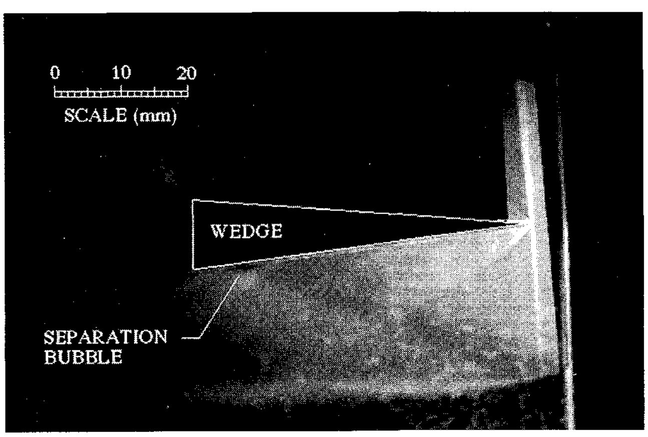

Fig. 2 Flow over a 5.8-deg half-angle wedge with Mach 2.0 nozzle.

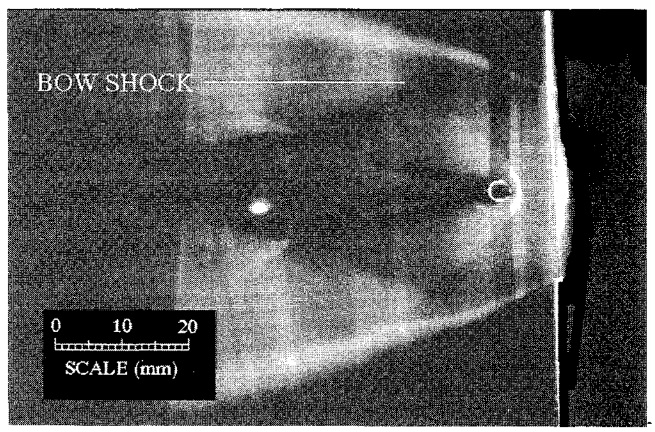

Fig. 3 Flow over a circular cylinder with Mach 1.5 nozzle.

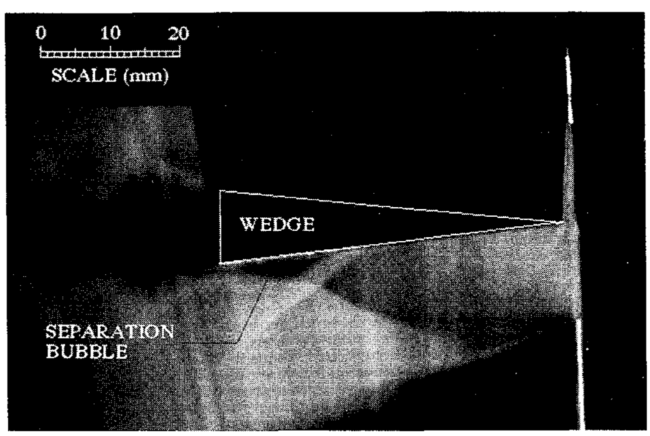

Fig. 4 Flow over a 5.8-deg half-angle wedge with Mach 1.5 nozzle.